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# Rigidity of Teichmüller curves

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Let  $f : V \rightarrow \mathcal{M}_g$  be a holomorphic map from a Riemann surface of finite hyperbolic volume to the moduli space of compact Riemann surfaces of genus  $g > 1$ . We say  $(V, f)$  is a *Teichmüller curve* if  $f$  is a local isometry for the Kobayashi metrics on domain and range. It is well-known that  $\mathcal{M}_g$  contains infinitely many Teichmüller curves.<sup>1</sup>

The purpose of this note is to show:

**Theorem 1** *Every Teichmüller curve  $f : V \rightarrow \mathcal{M}_g$  is rigid. Consequently  $V$  and  $f$  are defined over an algebraic number field.*

By *rigid* we mean any holomorphic deformation

$$f_t : V_t \rightarrow \mathcal{M}_g, \quad t \in \Delta,$$

with  $(V_0, f_0) \cong (V, f)$  (and  $V_t$  of finite volume) is trivial: we have  $(V_t, f_t) \cong (V, f)$  for all  $t$ .

**Proof.** The proof combines two facts:

1. If  $X, Y$  are two hyperbolic surfaces in the Teichmüller space  $\mathcal{T}_{h,n}$ , and the lengths of corresponding closed geodesics satisfy  $L(\gamma, X) \geq L(\gamma, Y)$  for all  $\gamma$ , then  $X = Y$ .
2. For a fixed finite-volume hyperbolic Riemann surface  $V$ , and a fixed integer  $g$ , there are only finitely many Teichmüller curves of the form  $f : V \rightarrow \mathcal{M}_g$ .

To see (1), observe that if the lengths of corresponding closed geodesics are the same, then the same is true for geodesic currents; in particular, the Liouville current  $\lambda_X$  (defined by the smooth invariant measure for the geodesic flow on  $X$ ) has the same length on both  $X$  and  $Y$ . But  $L(\lambda_X, S)$  is uniquely minimized at  $S = X$  [Wol], because of its strict convexity along

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<sup>1</sup>In fact square-tiled Riemann surfaces  $S$  are dense in  $\mathcal{M}_g$ , and any such  $S$  lies on a Teichmüller curve [HS, §1.5.2].

earthquake paths; thus  $X = Y$ . (The same principle is used in the proof of Nielsen realization problem [Ker].)

Alternatively, one can show the Lipschitz constant of an extremal stretch map from  $X$  to  $Y$  is controlled by ratios of lengths of geodesics, and hence  $X$  and  $Y$  are isometric [Th, Thm. 8.5].

To see (2), recall that the Kobayashi metric on  $V$  coincides with the hyperbolic metric (of constant curvature  $-4$ ), and the Kobayashi metric on  $\mathcal{M}_g$  coincides with the Teichmüller metric [Roy]. Moreover, a closed geodesic on  $\mathcal{M}_g$  is the unique loop of minimal length in its homotopy class [Bers], and there are only finitely many closed geodesics less than a given length [Iv]. Choose a pair of closed geodesics  $\alpha$  and  $\beta$  on  $V$  crossing at a point  $p$ . Since  $f$  is an isometry, there are only finitely many candidates for  $f(\alpha)$  and  $f(\beta)$ , and hence for  $f(p)$ . Consequently there are only finitely many candidates for the isometry  $f|_\alpha$ , and hence for the analytic map  $f$  itself.

Alternatively, one can note that (2) follows from the geometric Shafarevich conjecture: there are only finitely many nonconstant holomorphic maps  $f : V \rightarrow \mathcal{M}_g$  [Ar], [Par]. (Here  $f$  is assumed to locally lift to Teichmüller space  $\mathcal{T}_g \rightarrow \mathcal{M}_g$ .) For further discussion of this result, see e.g. [Mum], [Fal], or [Mc1].

Now consider a deformation  $f_t : V_t \rightarrow \mathcal{M}_g$  of a Teichmüller curve  $(V, f)$ . By the Schwarz lemma,  $f_t$  is distance-decreasing for all  $t$ . Let  $\gamma_0 \subset V_0$  be a closed geodesic. Since  $f_0$  is a local isometry,  $f_0(\gamma_0)$  is a Teichmüller geodesic, and hence of minimal length in its homotopy class. The corresponding geodesic  $\gamma_t$  on  $V_t$  therefore satisfies

$$L(\gamma_t, V_t) \geq L(f_t(\gamma_t), \mathcal{M}_g) \geq L(f_0(\gamma_0), \mathcal{M}_g) = L(\gamma_0, V_0).$$

Thus  $V_t \cong V_0$  by fact (1) above, and then  $f_t \cong f_0$  by fact (2); so  $(V, f)$  is rigid.

It is a standard fact that rigidity implies  $(V, f)$  is defined over a number field; otherwise the transcendental elements in its field of definition would give deformations. ■

**Finiteness.** A similar argument gives the following complementary result. Let us say  $f : X \rightarrow \mathcal{M}_g$  is *generalized Teichmüller curve* if  $X$  is a hyperbolic Riemann surface (possibly of infinite area), and  $f$  is a holomorphic, generically 1-1 local isometry for the Kobayashi metric.

**Proposition 2** *For a fixed genus  $g$  and  $L > 0$ , there are only finitely many generalized Teichmüller curves  $f : X \rightarrow \mathcal{M}_g$  such that  $X$  has a closed geodesic  $\gamma$  of length  $\leq L$ .*

**Proof.** There are only finitely many closed geodesics of length  $\leq L$  in  $\mathcal{M}_g$ , so there are only finitely many possibilities for  $\delta = f(\gamma)$ ; and  $\delta$  determines  $(X, f)$  up to isomorphism, by uniqueness of analytic continuation. ■

**Corollary 3** *For a fixed genus  $g$  and  $A > 0$ , there are only finitely many Teichmüller curves  $f : V \rightarrow \mathcal{M}_g$  with  $\text{area}(V) \leq A$ .*

**Proof.** An upper bound on the area of  $V$  gives an upper bound for the length of its shortest closed geodesic. ■

A related proof, and additional finiteness results, appear in [SW].

**Remark: curves in  $\mathcal{A}_g$ .** By composing with the map  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  sending a curve to its Jacobian, every Teichmüller curve also determines a curve

$$Jf : V \rightarrow \mathcal{A}_g$$

in the moduli space of Abelian varieties.

These curves are generally not rigid, even when  $Jf$  is an isometry for the Kobayashi metric. Indeed, Möller has given an example in the case  $g = 3$  where every  $X \in f(V)$  covers a fixed elliptic curve  $E_0$ , and consequently  $Jf(v)$  is isogenous to  $B(v) \times E_0$  for all  $v \in V$  [Mo2, §3]. Thus the curve  $Jf : V \rightarrow \mathcal{A}_g$  can be deformed by varying the factor  $E_0$ . Rigidity for  $Jf$  under some additional hypotheses follows from [Mo1, Thm 5.1] (see also [Mo3, Cor. 6.2], which bridges a gap in the original proof).

**Notes and references.** For more on Teichmüller curves, their connection to polygonal billiards, and their relation to the horocycle and geodesic flows over moduli space, see e.g. [V], [KS], [MT], [Mc2], [Mo1], [Mo2], [BM].

## References

- [Ar] S. Arakelov. Families of algebraic curves with fixed degeneracies. *Math. USSR Izv.* **35**(1971), 1269–1293.
- [Bers] L. Bers. An extremal problem for quasiconformal maps and a theorem by Thurston. *Acta Math.* **141**(1978), 73–98.
- [BM] I. I. Bouw and M. Möller. Teichmüller curves, triangle groups and Lyapunov exponents. *Preprint, 2006*.
- [Fal] G. Faltings. Some historical notes. In G. Cornell and J. H. Silverman, editors, *Arithmetic Geometry*, pages 1–8. Springer-Verlag, 1986.

- [HS] P. Hubert and T. A. Schmidt. An introduction to Veech surfaces. In *Handbook of Dynamical Systems. Vol. 1B*, pages 501–526. Elsevier, 2006.
- [Iv] N. V. Ivanov. Coefficients of expansion of pseudo-Anosov homeomorphisms. *J. Soviet Math.* **52**(1990), 2819–2822.
- [KS] R. Kenyon and J. Smillie. Billiards on rational-angled triangles. *Comment. Math. Helv.* **75**(2000), 65–108.
- [Ker] S. Kerckhoff. The Nielsen realization problem. *Ann. of Math.* **177**(1983), 235–265.
- [MT] H. Masur and S. Tabachnikov. Rational billiards and flat structures. In *Handbook of Dynamical Systems, Vol. 1A*, pages 1015–1089. North-Holland, 2002.
- [Mc1] C. McMullen. From dynamics on surfaces to rational points on curves. *Bull. Amer. Math. Soc.* **37**(2000), 119–140.
- [Mc2] C. McMullen. Dynamics of  $SL_2(\mathbf{R})$  over moduli space in genus two. *Annals of Math.* **165**(2007), 397–456.
- [Mo1] M. Möller. Variations of Hodge structures of a Teichmüller curve. *J. Amer. Math. Soc.* **19**(2006), 327–344.
- [Mo2] M. Möller. Shimura- and Teichmüller curves. *Preprint, 2005*.
- [Mo3] M. Möller and E. Viehweg. Kobayashi geodesics in  $\mathcal{A}_g$ . *In preparation*.
- [Mum] D. Mumford. *Curves and their Jacobians*. University of Michigan Press, 1975.
- [Par] A. N. Parshin. Algebraic curves over function fields. *Soviet Math. Dokl.* **183**(1968), 524–526.
- [Roy] H. L. Royden. Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces*, pages 369–384. Princeton University Press, 1971.
- [SW] J. Smillie and B. Weiss. Finiteness results for flat surfaces: Large cusps and short geodesics. *Preprint, 2008*.

- [Th] W. P. Thurston. Minimal stretch maps between hyperbolic surfaces. *Preprint, 1998*.
- [V] W. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. math.* **97**(1989), 553–583.
- [Wol] S. Wolpert. Thurston’s Riemannian metric for Teichmüller space. *J. Diff. Geom.* **23**(1986), 143–174.

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